2+1 Black Hole with a Dynamical Conical Singularity

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Abstract

We find black hole solutions to Euclidean 2+1 gravity coupled to a relativistic particle which have a dynamical conical singularity at the horizon. These solutions mimic the tree level contribution to the partition function of gravity coupled to a quantum field theory. They are found to violate the standard area law for black hole entropy, their entropy being proportional to the total opening angle. Since each solution depends on the number of windings of the particle path around the horizon, the significance of their summation in the path integral is considered.

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1 Introduction

The origin of the entropy of black holes is a problem which focused a large amount of work, ever since the formulation of the laws of black hole thermodynamics [1]. Despite the difficulties raised by the question of what the statistical mechanical interpretation of black hole entropy is, it is well known how to compute it by thermodynamical techniques. The entropy arises from the tree level contribution of the Euclidean gravitational action to the partition function [2], and is essentially due to the presence of a horizon. It bears no direct relation to the fluctuations of any external (quantum) field and is therefore an intrinsic characteristic of the black hole.

Thus far, there have been several attempts to give an interpretation of black hole entropy in terms of microscopic degrees of freedom. Most recently, promising results have come from string theory [3] where counting of elementary and solitonic states agrees with the area law for black hole entropy (which can be generalized to any dimension):

$$S = \frac{A_h}{4G},\tag{1}$$

where A_h is the (generalized) area of the horizon and G is the Newton constant. However, string theory does not provide yet a general explanation of this formula; rather, it has to be checked case by case, and indeed it has been checked only in a few special cases. In a different direction, and in the context of 2+1 dimensional black holes, there was also the recent proposal of Ref. [4].

Yet another direction is to consider examples for which the area law (1) is corrected or violated . A celebrated example where such a violation occurs is the extremal black hole which, according to Refs. [5, 6], has vanishing entropy (see, however, Ref. [7] for a string theory-based confutation of this claim). In a recent work, Englert, Houart and Windey [8] have found another interesting example: a dynamical conical singularity produced by a Nambu-Goto string wound around the horizon decreases the entropy of the black hole proportionally to the deficit angle. In Refs. [9, 10] a similar mechanism was used to shift the free energy of the black hole in models in which the string action arises as the effective action of a vortex in a spontaneously broken gauge theory.

In this paper we will focus on 2+1 dimensional anti-de Sitter black holes [11]. The reason is twofold: 2+1 dimensional gravity is simpler to handle than in 3+1 dimensions, especially when one is dealing with matter consisting of point particles; BTZ black holes have a nicer thermodynamical behaviour than Schwarzschild ones. This is seen as follows. Schwarzschild black holes have a negative heat capacity and hence an unstable behaviour; their correct thermodynamical interpretation is in terms of sphalerons [12, 13]. The 2+1 black holes on the other hand have a positive heat capacity which allows us to consider them a well-defined thermodynamical system, characterized e.g. by a real partition function.

We will use the path integral approach to black hole thermodynamics, where

the following expression for the canonical partition function is used:

$$Z(\beta) = \int \mathcal{D}g\mathcal{D}\phi \; e^{-I_{grav}[g]-I_{mat}[g,\phi]}. \tag{2}$$

β is the period of the Euclidean time and is identified with the inverse of the temperature. For the picture to be consistent, one has to integrate over periodic Euclidean metrics and matter configurations, the actions I_{grav} and I_{mat} also being the Euclidean gravitational and matter actions. Going to imaginary time also fixes the topology of the Euclidean black hole manifold: generically in d dimensions the Euclidean topology of a black hole is $R^2 \times S^{d-2}$; the R^2 factor includes the possibility of a conical singularity. This topology gives to the black hole a non vanishing horizon area, thus disconnecting black hole solutions from (hot) flat space.

It is known that one can reproduce all the n-point functions of a quantum field theory from the path integral of the action of a relativistic particle (see for instance Ref. [14]). To be more specific, one can take a particle of mass m and action:

$$I_{part} = mL(path) = m \int ds = m \int \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \; ds \tag{3}$$

to mimic the quantum field theory of a real massive scalar field. The vacuum fluctuations of the free field will be described by paths of S^1 (loop) topology. These loops are necessarily off-shell in the Lorentzian picture, so when going to imaginary time there are no restrictions of causal type on them. Hence, to calculate the partition function, one sums over all closed Euclidean paths of the relativistic particle. The expression (2) is thus equivalent to:

$$Z(\beta) = \int \mathcal{D}g\mathcal{D}x(s) \; e^{-I_{grav}[g]-I_{part}[g,x(s)]}. \tag{4}$$

Now the problem of coupling particle matter to 2+1 gravity is also well-known [15]: the only back reaction caused by the particles is to create a conical singularity along their paths. The main scope of this paper is to show, treating the matter action as in (4), that the matter indeed contributes to the tree level entropy of the black hole, by a mechanism which includes a dynamical conical singularity. This contribution leads to a decrease of the purely gravitational value of the entropy, thus correcting the law (1).

The paper is organized as follows: in section 2 we find a classical solution to Euclidean 2+1 gravity coupled to a relativistic particle; this solution represents macroscopical topologically non-trivial fluctuations of the matter field; we also derive an equation of motion for the deficit angle. In section 3 we analyse the thermodynamics of the black hole with the dynamical conical singularity and we derive the corrected area law for entropy. In the concluding section we try to provide a link with results in other dimensions [8], and we briefly speculate on how to consider at the same time the different solutions corresponding to a different winding number.

2   An equation for the deficit angle

The action of Euclidean 2+1 gravity coupled to a relativistic particle is the following:

$$I = I_{grav} + I_{part}. \tag{5}$$

In the Euclidean section, the black hole has topology $R^2 \times S^1$, including the possibility of a deficit angle in the R^2 factor. As a consequence of this topology, there is a point in the R^2 factor for which the S^1 circle is of minimal size. This submanifold is identified with (the analytic continuation to imaginary time of) the horizon.

Thus, the closed paths divide naturally into homotopy classes, characterized by how many times the path winds around the S^1 factor. The condition for a trajectory to belong to a particular class is:

$$\Delta_{tot}\phi \equiv \int_{path} d\phi = 2\pi k; \tag{6}$$

ϕ is the variable labelling the S^1 factor; we allow for both signs of k.

The minimum of I_{part} is clearly attained, for a particle moving according to (6), when the path is wound around the minimal circle at the horizon; homotopically disjoint solutions are obtained choosing different integer values of k. These trajectories seem tachyonic at a first look, but we have to consider them on the same footing as any other since we are computing a partition function. They are macroscopic vacuum configurations due to the non-trivial topology of the spacetime of the (Euclidean) black hole. It has also to be noted that these are the only closed path classical solutions to the geodesic equations, and this will be the reason why they contribute to the partition function.

Once the matter equations of motion are solved, the matter action (3) takes the simple form:

$$I_{part} = m|k|L_h, \tag{7}$$

where L_h is the circumference of the horizon.

Let us now consider the gravitational action and the back-reaction of the matter on the geometry. The gravitational action I_{grav} is taken to be covariant and supplemented by local boundary terms at infinity:

$$I_{grav} = -\frac{1}{16\pi G} \int d^3x \sqrt{g} \left(\mathcal{R} + \frac{2}{l^2} \right) + B_\infty. \tag{8}$$

The cosmological constant $\Lambda = -1/l^2$ is negative to incorporate BTZ black hole solutions which are asymptotically anti-de Sitter [11].

An easy way to derive the equations of motion and the on-shell value of the gravitational action is to start from the hamiltonian action, which is vanishing on shell for a static black hole. For a static, circularly simmetric metric, which we take here for simplicity, we have:

$$ds^2 = N^2 dt^2 + F dr^2 + R^2 d\phi^2 \tag{9}$$

and the hamiltonian action reduces to:

$$I_{ham} = \int dt \; d^2x N\mathcal{H}, \tag{10}$$

where $\mathcal{H}$ is the hamiltonian constraint.

Now, to render this action covariant at the horizon (i.e. to avoid a 'false' boundary term at the horizon in the variation of the action), one has to add a local boundary term there [16]. This term turns out to be $-\frac{1}{4G}L_h$.

When turning to analyze the boundary term at infinity in the variation of I_{ham}, we find that the equations of motion are obtained when δM is put to 0, M being the ADM mass of an asymptotically anti-de Sitter space-time, as defined[1] in [17]. One then calls this covariant action I_{micro} with reference to the thermodynamical microcanonical ensemble in which the energy of the system is fixed [18]:

$$I_{micro} = I_{ham} - \frac{1}{4G}L_h. \tag{11}$$

However, it is useful to have an action suitable for fixing β, the periodicity of the Euclidean time t, instead of M. As these two variables are canonically conjugated [11], it is straightforward to obtain it:

$$I_{can} = I_{micro} + \beta M, \tag{12}$$

this being the canonical action since now the equations of motion are obtained by fixing β, i.e. by fixing the temperature as in the canonical ensemble. Note that βM is a local boundary term at infinity.

We turn now to solving the equations of motion for the metric. The bulk of the variation of the gravitational action yields simply the equations of pure 2+1 gravity with a negative cosmological constant. We pick the BTZ black hole solution, which is (9) with:

$$N^2 = F^{-1} = \frac{r^2}{l^2} - 8GM, \qquad R = r. \tag{13}$$

The 'false' boundary term at the horizon of the variation of I_{grav}, which is the same for I_{micro} and I_{can}, is [16]:

$$\delta I_{grav}|_{hor} = \frac{1}{4G}\left(\frac{\beta}{\beta_H} - 1\right)\delta L_h, \tag{14}$$

where β_H is the inverse Hawking temperature, a function of the metric parameters (say M):

$$\frac{2\pi}{\beta_H} = \left.\frac{N'}{\sqrt{F}}\right|_{r=r_h} \equiv \kappa \tag{15}$$

and $L_h = 2\pi R_h$ is the size of the horizon.

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[1] A subtraction term is needed in the definition. We choose the vacuum black hole (i.e. the solution corresponding to $M = 0$) as the reference space-time.

Once the matter equations of motion are solved, the variation of the matter action (7) with respect to the gravitational degrees of freedom reads:

$$\delta_{(grav)} I_{part} = m|k|\delta L_h. \tag{16}$$

Combining (14) and (16) and the requirement that L_h should not be fixed in advance (i.e. demanding covariance), the condition $\delta I = 0$ yields, for the horizon contribution, the equation of motion for the (dynamical) deficit angle:

$$\left(1 - \frac{\beta}{\beta_H}\right) = 4G|k|m. \tag{17}$$

We can rearrange this equation to obtain a condition on the periodicity β:

$$\beta = \beta_H(1 - 4G|k|m). \tag{18}$$

This is the only effect of the matter on the geometry. Notice that $\mathcal{R}$ is then singular at $r = r_h$, but this singularity is generated by the matter distribution, so no conceptual problem is arising.

Note that we could have straightforwardly inserted the stress–energy tensor of the relativistic particle into the Einstein equations finding a conical singularity along the particle's path, as in the work of Deser, Jackiw and 't Hooft [15]. We preferred instead to give an alternative derivation in a formalism that yields directly an equation for the deficit angle, which measures the strength of the conical singularity.

We will not consider the case for which $\beta < 0$, i.e. when the deficit angle exceeds 2π, because it is unphysical when only one particle is present [15]. Also, the case for which $\beta = 0$ is a degenerate one because the topology turns from the cone to the cylinder, ceasing to be a black hole topology; we will not consider it either.

We now turn to the thermodynamics, which can be derived either from I_{micro} or from I_{can}.

3 The thermodynamics

To describe the thermodynamical behaviour of 2+1 black holes we use the path integral representation of the partition function. In this formalism, the black hole thermodynamics has a clear quantum mechanical origin (i.e. the $\hbar$ dependence of the quantities is naturally fixed). We consider in the following the semiclassical approximation in which we retain only the leading order of a saddle point (tree level approximation).

Let us first calculate the temperature[2] of the solution found in the preceding

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[2] One has to spend a few words on the thermodynamical conjugate variables β and M. Since the geometry is asymptotically anti-de Sitter, these two quantities do not coincide with the local inverse temperature and the energy at infinity. This is due to the fact that the Killing vector $\frac{\partial}{\partial t}$ has an infinite norm at infinity. Rather, β can be thought of as the global inverse temperature; its thermodynamical conjugate is M and can moreover be identified with the ADM mass [11, 17].

section. For a BTZ black hole, eqs. (15) and (18) give:

$$\beta = \frac{\pi l}{\sqrt{2GM}}(1 - |k|\eta), \quad \eta = 4Gm. \tag{19}$$

This relation entails that, for $|k|\eta < 1$ (the physically relevant case), the temperature $T \sim \sqrt{M}$. This implies that the BTZ black hole has a positive heat capacity, because $c_V = \frac{\partial M}{\partial T} \sim T > 0$. This behaviour is in sharp contrast with the case of the 4-dimensional Schwarzschild black hole, which has a negative heat capacity and hence an unstable thermodynamic behaviour. In our case, the positivity of c_V will allow us to use a well defined (real) canonical partition function.

The standard way of calculating the canonical partition function is to evaluate (2) taking I_{grav} to be I_{can}: we integrate over all Euclidean metrics of fixed period β [2].

There is an alternative derivation [18] (see also [16]) that uses I_{micro} instead of I_{can}. The path integral using this action yields directly the exponential of the entropy, i.e. the microcanonical partition function or the density of states:

$$e^{S(M)} = \int \mathcal{D}g \mathcal{D}x(s) \, e^{-I_{micro}[g] - I_{part}[g,x(s)]}. \tag{20}$$

We integrate over the Euclidean metrics which are periodic in the Euclidean time and for which the ADM mass is fixed to M.

Evaluated by the saddle point approximation to the leading order and for a fixed value of the winding number k, (2) gives:

$$\begin{aligned}
-\ln Z_k(\beta) &\simeq & (I_{can} + I_{part})|_{on-shell} \\
&=& \beta M - \frac{L_h}{4G} + m|k|L_h \\
&=& \beta M - \frac{L_h}{4G}(1 - |k|\eta).
\end{aligned} \tag{21}$$

One has to be careful at this stage because M and L_h, as given by the ADM mass formula and by $L_h = 2\pi R_h$, are functions of β_H rather than β; when this is taken into account, $\ln Z_k(\beta)$ has a nonlinear dependence on the winding number k:

$$\ln Z_k(\beta) = \frac{\pi^2 l^2}{2G\beta}(1 - |k|\eta)^2. \tag{22}$$

The entropy is then obtained from the standard thermodynamical relation:

$$S(M) = -\left(\beta\frac{\partial}{\partial\beta} - 1\right)\ln Z(\beta)\bigg|_{\beta=\beta(M)}, \tag{23}$$

where $\beta = \beta(M)$ is obtained by inverting the equation $M = -\frac{\partial}{\partial\beta}\ln Z(\beta)$. The result is thus:

$$S_k = \frac{L_h}{4G}(1 - |k|\eta) = \pi l\sqrt{\frac{2M}{G}}(1 - |k|\eta). \tag{24}$$

This is exactly the same result that we could have obtained by use of (20) and (11), a consequence of the equivalence of the two approaches when the contribution of only one saddle point is taken into account. Since S_k is a function of M, it has a linear dependence on k.

The effect of the dynamical conical singularity on the thermodynamics of the black hole is hence the following: with respect to the $k = 0$ case, the entropy is lowered and the free energy $F_k(\beta) = -\beta^{-1} \ln Z_k(\beta)$ is increased. Hence, the black hole topology allows for nontrivial configurations of the matter fields, here represented by a relativistic particle, which lead to a correction of the law (1) for black hole entropy already at tree level.

As a concluding remark on the thermodynamical behaviour of our solution, notice that if we want Z_k to be represented by a Laplace transform of the density of states e^{S_k}, we need a condition on the temperature. Indeed, the integral:

$$Z_k(\beta) = \int_0^\infty dM \ e^{-\beta M} e^{S_k(M)} \tag{25}$$

can be calculated by saddle point approximation only if the following condition holds, assuring that the integral can be approximated by a gaussian integral:

$$\left| \frac{\partial^2 S_k}{\partial M^2} \right|_{M=M_{s.p.}}^{-\frac{1}{2}} \ll M_{s.p.}, \tag{26}$$

where $M_{s.p.}$ is the value of M at the saddle point, expressed in terms of β. The condition (26) becomes then:

$$\beta \ll \frac{\pi^2 l^2}{4G} (1 - |k|\eta)^2, \tag{27}$$

which means high temperature.

4   Concluding remarks

Let us first put forward the conclusions of the preceding sections. We have shown that a 2+1 black hole with a particle path wound around its horizon violates the (three-dimensional) standard area law for its entropy. Indeed, in the presence of a particle the topological R^2 factor of the black hole manifold has in fact the shape of a cone near the horizon. The entropy of this black hole is simply proportional to the total opening angle.

This result is very similar to the 3+1 dimensional computation of [8], independently of the different thermodynamics of BTZ and Schwarzschild black holes. However, the physical interpretation behind the effect is slightly different.

In 2+1 dimensions, the worldline of the particle is the representation of a macroscopic vacuum fluctuation. It is on this ground that is has to be included in the computation of the partition function. On the other hand, in 3+1 dimensions one is faced with a string configuration whose worldsheet has the topology of a

sphere, which does not represent in (closed) string theory conventional vacuum fluctuations; hence, in this case one is handling a more complicated phenomenon.

One could try to extend this result to dimensions higher than 4. Generically, one can guess that the same effect can arise for a d dimensional black hole when the Euclidean worldvolume of a $(d-3)-$brane, of S^{d-2} topology, is wrapped around the horizon. Still, the physical interpretation of such a worldvolume should be made clear.

Going back to the 2+1 dimensional case, one can make some further considerations.

Whether the result obtained is a true violation of the area law (1) or not depends on the stability of a solution at fixed winding number k. If the lifetime of such a solution is long enough to establish a thermodynamic equilibrium, then the entropy of this configuration has a physical significance. If on the other hand it appears that this solution is highly unstable and decays rapidly to another configuration, then this result is not of great relevance. Indeed, the $k = 0$ solution is the one of the lowest action[3] and free energy.

These last considerations lead us to considering the more general problem where all the solutions are taken into account, i.e. when there are no more restrictions on the winding number k. Since a solution exists for each winding number k, then in the path integral (2) or (20) the integrand has several minima. One is tempted to sum over them, evaluating each one by a saddle point approximation [9, 10].

However, since our aim is to calculate thermodynamical averages and not quantum expectation values, this procedure is not straightforward. Indeed, all the standard thermodynamical relations are retrieved from the saddle point approximation of the Laplace transform (25) or its inverse. There is no obvious reason for which this procedure still should work when the partition function or the density of states are derived from a path integral consisting of several saddle points. In particular, the microcanonical and the canonical approaches appear inconsistent in the general case. This problem, which goes beyond the framework of 2+1-dimensional black holes, is still under consideration.

Acknowledgments

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